CONVERGENCE RESULTS FOR SIMULTANEOUS & MULTIPLICATIVE DIOPHANTINE APPROXIMATION ON PLANAR CURVES.

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ABSTRACT. Let $\mathcal{C}$ be a non-degenerate planar curve. We show that the curve is of Khintchine-type for convergence in the case of simultaneous approximation in $\mathbb{R}^2$ with two independent approximation functions; that is if a certain sum converges then the set of all points (x, y) on the curve which satisfy simultaneously the inequalities $\|qx\| < \psi_1(q)$ and $\|qy\| < \psi_2(q)$ infinitely often has induced measure 0. This completes the metric theory for the Lebesgue case. Further, for multiplicative approximation $\|qx\|\|qy\| < \psi(q)$ we establish a Hausdorff measure convergence result for the same class of curves, the first such result for a general class of manifolds in this particular setup.

1. INTRODUCTION

Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a real, decreasing function. Throughout we will refer to ψ as an approximating function. Let $x \in \mathbb{R}$ and $\|x\|$ be the distance of x from $\mathbb{Z}$. That is, $\|x\| = \inf\{|x - z| : z \in \mathbb{Z}\}$. Further, if S is a (Lebesgue) measurable set in $\mathbb{R}^n$ then we will denote the Lebesgue measure, or more simply the measure, of S by $|S|_{\mathbb{R}^n}$.

Consider now the following system of n Diophantine inequalities

$$\|qx_i\| < \psi_i(q) \tag{1}$$

where $x_i \in \mathbb{R}$, $p_i \in \mathbb{Z}$, $q \in \mathbb{N}$ and $\psi_1, \psi_2, \ldots, \psi_n$ are approximation functions. Then a point $x \in \mathbb{R}^n$ is simultaneously $(\psi_1, \psi_2, \ldots, \psi_n)$-approximable if there are infinitely many q satisfying (1). The set of all such points $x \in \mathbb{R}^n$, will be denoted by $\mathcal{S}_n(\psi_1, \psi_2, \ldots, \psi_n)$.

Simultaneous approximation has another variant in the guise of multiplicative approximation; let $x \in \mathbb{R}^n$, then x is multiplicatively ψ-approximable if the inequality

$$\prod_{i=1}^{n} \|qx_i\| < \psi(q) \tag{2}$$

holds for infinitely many $q \in \mathbb{N}$. By analogy with the previous notion of Simultaneous approximation, we shall denote by $\mathcal{S}_n^*(\psi)$ the set of all multiplicatively ψ-approximable points x in $\mathbb{R}^n$.

The following results, the first of which is due to Khintchine and is a generalisation of Khintchine's own result of 1924 which deals with the case when $\psi_1 = \psi_2 = \cdots = \psi_n$. The second is due to Gallagher. Together they give an almost complete answer to the question of the 'size', in terms of n-dimensional Lebesgue measure, of $\mathcal{S}_n(\psi_1, \psi_2, \ldots, \psi_n)$ and $\mathcal{S}_n^*(\psi)$.

Theorem K. *Let* $\psi_1, \psi_2, \ldots, \psi_n$ *be approximation functions as defined above. Then*

$$|\mathcal{S}_n(\psi_1, \psi_2, \ldots, \psi_n)|_{\mathbb{R}^n} = \begin{cases} 0 & \text{if} \quad \sum \psi_1(h) \ldots \psi_n(h) < \infty \\ FULL & \text{if} \quad \sum \psi_1(h) \ldots \psi_n(h) = \infty \end{cases}$$

Theorem G. *Let* ψ *be an approximating function. Then*

$$|\mathcal{S}_n^*(\psi)|_{\mathbb{R}^n} = \begin{cases} 0 & \text{if} \quad \sum \psi(h)^n \log^{n-1} h < \infty \\ FULL & \text{if} \quad \sum \psi(h)^n \log^{n-1} h = \infty \end{cases}$$

It is to be understood that the term 'full' means that the complement of the set in question is of measure 0.

In Theorem(s) K and G the approximation problem is an independent variables problem; no functional relationship exists between any of the coordinates of x. Once a functional relationship is assumed to exist , that is the points x are constrained to lie on a sub-manifold $\mathcal{M} \subset \mathbb{R}^n$, then the corresponding approximation problems become a great deal more difficult. As we shall see below, until very recently hardly any general results at all were known for such problems. Furthermore, of those results that have been established, most hold only for planar curves with sufficient curvature conditions. There is next to nothing known for manifolds of dimension in $\mathbb{R}^n$ where $n \geq 3$. This is in stark contrast to the dual form of approximation; see for example [1], [4] or [7], where the state of knowledge is much more complete. Before we discuss some of the more significant results which are known to hold in the so-called "dependent variables"[1] case, it is necessary to define the concept of non-degeneracy.

Let $C^{(m)}(U)$ be the space of all m–continuously differentiable functions f where $f : U \to \mathbb{R}$ with U being an open set in $\mathbb{R}^n$. A map $g : U \to \mathbb{R}$ is said to be non-degenerate at $u \in U$ if there exists some $l \in \mathbb{N}$ such that $g \in C^{(l)}(B(u, \delta))$ for some sufficiently small $\delta > 0$ with $B(u, \delta) \subset U$ and the partial derivatives of g evaluated at u span $\mathbb{R}^n$. The map g is non-degenerate if it is non-degenerate at almost all points $u \in U$. Let $\mathcal{M}$ be a sub-manifold of $\mathbb{R}^n$. Then $\mathcal{M}$ is said to be non-degenerate if $\mathcal{M} = g(U)$ where g is non-degenerate. The geometric interpretation of non-degeneracy is that the manifold is curved enough that it deviates from any hyperplane.

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[1]the terminology is due to Sprindžuk [14]

Non-degeneracy is not a particularly restrictive condition and a large class of manifolds satisfy this condition.

Note that if the topological dimension, $\dim \mathcal{M}$, of the manifold is strictly less than n then $|\mathcal{M}|_{\mathbb{R}^n} = 0$. As we wish to make measure theoretic statements about points that lie on the manifold we work with the induced measure, $|\cdot|_{\mathcal{M}}$. All "zero-full" statements are made with respect to this restricted measure.

Schmidt [13] proved one of the first major results that attempted to address the problem of generalising Theorems K and G to manifolds.

Theorem S. *Let $\psi : \mathbb{R}^+ \to \mathbb{R} : x \mapsto x^{-\tau}$ where $\tau > 0$ and $2\tau > 1$. Then for any $C^{(3)}$ non-degenerate planar curve, $\mathcal{C}$,*

$$|\mathcal{C} \cap \mathcal{S}_2(\psi, \psi)|_{\mathcal{C}} = 0.$$

Note that this result is not quite a Khintchine-type result for convergence as the measure 0 statement does not depend on the convergence of an associated sum. This question was settled very recently for an arbitrary approximation function ψ. In [3], Beresnevich, et. al. established the divergence part of such a theorem by showing that for any $C^{(3)}$ non-degenerate curve

$$|\mathcal{C} \cap \mathcal{S}_2(\psi, \psi)|_{\mathcal{C}} = \text{FULL} \quad \text{if} \quad \sum_{h=1}^{\infty} \psi^2(h) = \infty. \tag{3}$$

To prove this result the authors adapted their notion of "local ubiquity" as developed in [2]. Interestingly the convergence case initially alluded them. This is yet another manifestation of the difficulties one encounters when trying to establish dependent variable analogues of many of the classical results of Diophantine approximation. In the classical setting the convergence case is usually straightforward and all the substance is in the divergence case. However, for dependent variable problems it turns out that both halves of a Khintchine-type result are highly non-trivial with the convergence case sometimes turning out to be the more difficult of the two.

In the above case the main obstacle to establishing the convergence statement was the need for precise information about the number of rational points near the curve. This is a notoriously difficult problem and the best known result, due to Huxley [10, Theorem 4.2.4], was good enough to ensure a reasonable distribution of rational points near the curve for the application of local ubiquity, but gave estimates which were too large for certain sums when considering the convergence case. Using a result of Vaughan's, which appeared as an appendix in [3] and was a significant sharpening of Huxley's result for the class of rational quadrics, curves that are the image of the unit circle, the parabola $\{(x, y) : y = x^2\}$ or the hyperbola $\{(x, y) : x^2 - y^2 = 1\}$ under a rational affine transformation of the plane, the authors were able to establish the convergence counterpart of this theorem for such manifolds. Subsequently Beresnevitch & Velani [5] were able to establish a zero–full result for simultaneous approximation with different approximation functions,

Theorem BV 1 below, and a convergence result for multiplicative simultaneous approximation, Theorem BV 2, for the class of non-degenerate rational quadrics.

Theorem BV 1. *Let ψ_1, ψ_2 be approximating functions and $\mathcal{Q}$ a $C^{(3)}$ non-degenerate rational quadric. Then*

$$|\mathcal{Q} \cap \mathcal{S}_2(\psi_1, \psi_2)|_{\mathcal{Q}} = \begin{cases} 0 & \text{if} \quad \sum_{q=1}^{\infty} \psi_1(q)\psi_2(q) \leq \infty, \\ FULL & \text{if} \quad \sum_{q=1}^{\infty} \psi_1(q)\psi_2(q) = \infty. \end{cases}$$

Theorem BV 2. *Let ψ be an approximating function and $\mathcal{Q}$ a $C^{(3)}$ non-degenerate rational quadric. Then*

$$|\mathcal{Q} \cap \mathcal{S}_2^*(\psi)|_{\mathcal{Q}} = 0 \quad \text{if} \quad \sum_{q=1}^{\infty} \psi_1(q)\psi_2(q) < \infty.$$

Of particular note is Theorem BV 2, which was the first result of its type for multiplicative simultaneous approximation on a reasonably general class of manifolds.

The counterpart convergence statement to (3) was finally established in [15]. To do so, Vaughan & Velani managed to extend Vaughan's result to any sufficiently smooth function. This result is crucial to our arguments below and we shall postpone giving the full statement until §3 when we can put the result into a more clearer context.

It should be noted that whilst Beresnevitch & Velani were able only to establish the convergence part of Khintchine's theorem for rational quadrics, they did prove the divergence part in full generality.

Theorem BV 3. *Let ψ_1, ψ_2 be approximating functions and $\mathcal{C}$ a $C^{(3)}$ non-degenerate planar curve. Then*

$$|\mathcal{C} \cap \mathcal{S}_2(\psi_1, \psi_2)|_{\mathcal{C}} \quad \text{is} \quad FULL \quad \text{if} \quad \sum_{q=1}^{\infty} \psi_1(q)\psi_2(q) = \infty.$$

In §3 we establish the convergence case for Theorem BV3 thus completing the Lebesgue metric theory for this particular problem.

The results discussed above are among the few that are known to hold for any reasonably general class of manifolds. Obviously there is still a great deal to be done before the simultaneous theory is anywhere near as complete as the dual case. A particularly significant result is due to D. Kleinbock & G. Margulis [11], who established the validity of the Baker-Sprindžuk conjecture:

Theorem KM. *Let $\mathcal{M}$ be a non-degenerate manifold in $\mathbb{R}^n$ and $\psi : \mathbb{R}^+ \to \mathbb{R}^+ : x \mapsto x^{-\tau}$. If $\tau > 1$ then*

$$|\mathcal{M} \cap \mathcal{S}_n^*(\psi)|_{\mathcal{M}} = 0.$$

Theorem KM gives us hope that it will be possible to establish the analogues of the classical results of Khintchine and Gallagher for higher dimensional manifolds in much the same way that these results exist for the dual case. However, there is still a long way to go before this becomes reality.

2. STATEMENT OF RESULTS

Let I be some open interval in $\mathbb{R}$ and $f \in C^{(3)}(I)$ such that for all $x \in I$:

(1) there exist constants $c_1 > c_2 > 0$ with $c_1 > f'(x) > c_2$,
(2) $f'' \neq 0$.

Under these assumptions it is readily verified that the curve C_f, where

$$C_f := \{(x, f(x)) : x \in I\},$$

is non-degenerate. We shall be assuming these conditions throughout the remainder of this article.

In [15], the authors conjectured that

$$|\mathcal{C}_f \cap \mathcal{S}_2(\psi_1, \psi_2)|_{\mathcal{C}_f} = 0 \quad \text{if} \quad \sum_{h=1}^{\infty} \psi_1(h)\psi_2(h) < \infty \tag{4}$$

and

$$|\mathcal{C}_f \cap \mathcal{S}_2^*(\psi)|_{\mathcal{C}_f} = 0 \quad \text{if} \quad \sum_{h=1}^{\infty} \psi(h) \log h < \infty. \tag{5}$$

The conjecture stated in (5) is a special case of Theorem 1 below. However, before stating the theorem it is necessary to briefly discuss Hausdorff s-measures.

Lebesgue measure is in some sense a relatively "coarse" measure of the size of a set. The notion of measure 0 can hide a multitude of finer structure; structure that is often of great interest in Diophantine approximation. To overcome this technical deficiency, one can use the idea of Hausdorff s-measure and the related notion of Hausdorff dimension. Hausdorff s-measures and dimensions are (theoretically) computable quantities that give more precise information about sets of Lebesgue measure 0 and it is for this reason that they play such a central role in Diophantine approximation. We outline only the very basics of the theory of Hausdorff measures. For a more detailed exposition of the theory and its many applications in mathematics, see either of the excellent books by Falconer ([8] or [9]).

Let $X \subset \mathbb{R}^n$ and $s \geq 0$. For any $\delta > 0$, a δ–cover, $\mathcal{C}_\delta(X)$, of X is a countable collection of balls B_i such that $X \subset \bigcup B_i$ and $\text{diam} B_i \leq \delta\}$. The set function $\mathcal{H}_\delta^s(\cdot)$, where

$$\mathcal{H}_\delta^s(X) := \inf \left\{ \sum \text{diam}^s B_i \right\}$$

with the infimum taken over all δ-covers of X, is an outer measure. Taking the limit of this quantity as $\delta \to 0$ gives the Hausdorff s-measure of X. That

is,

$$\mathcal{H}^s(X) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(X).$$

When s takes on values in $\mathbb{N} \cup \{0\}$ then $\mathcal{H}^s$ coincides with s-dimensional Lebesgue measure. One of the most useful properties of Hausdorff s-measure is the existence of a unique value of s, which we shall denote by $\dim X$, where the s-measure jumps from 0 to ∞ as the parameter s passes through the value $\dim X$ from right to left. More precisely,

$$\dim X := \inf\{s \in \mathbb{R}^+ : \mathcal{H}^s(X) = 0\} = \sup\{s \in \mathbb{R}^+ : \mathcal{H}^s(X) = \infty\}.$$

It is exactly this quantity that we refer to as the Hausdorff dimension of X. At the critical exponent $s = \dim(X)$, the value of $\mathcal{H}^{\dim(X)}(X)$ can be 0, > 0, or ∞. Sets X where $\mathcal{H}^s(X) \in (0, \infty)$ are known as s-sets with probably the most famous example of am s-set whose Lebesgue measure is 0 being the classical middle-thirds Cantor set $\mathbb{K}$. Indeed, t is well known that $\dim \mathbb{K} = \ln 2 / \ln 3$ and that $\mathcal{H}^{\ln 2 / \ln 3}(\mathbb{K}) = 1$. Thus the s-measure of a set can be used to garner more precise information about Lebesgue measure 0 and this is one of their principle uses in metric Diophantine approximation.

We now come to Theorem 1, which is the Hausdorff s-measure version of Theorem BV 3.

Theorem 1. *Let ψ be an approximating function and $0 < s \leq 1$. Then*

$$\mathcal{H}^s(C_f \cap S_2^*(\psi)) = 0 \quad if \quad \sum_{h=1}^{\infty} h^{1-s}(\log^s h)\psi^s(h) < \infty.$$

Note that as $\mathcal{H}^1$ coincides with 1-dimensional Lebesgue measure and as we are working with the induced measure on the manifold, in this case a 1-dimensional manifold, Conjecture 5 follows immediately as a special case of Theorem 1. For the cases when $0 < s < 1$ Theorem 1 appeared as a conjecture in [15].

Furthermore, the proof of Theorem 1 can be adapted to settle claim 4 and it is exactly this result that we present as Theorem 2 below.

Theorem 2. *Let ψ_1, ψ_2 be approximating functions. Then*

$$|C_f \cap S_2(\psi_1, \psi_2)|_{C_f} = \left\{ 0 \quad if \quad \sum_{h=1}^{\infty} \psi_1(h)\psi_2(h) < \infty. \right.$$

Note that as mentioned above, in §1, this establishes the convergence counterpart to Theorem BV 3 and this completes the metric theory, at least in the case of Lebesgue.

3. PROOF OF THEOREM 1

We are given that

$$\sum_{h=1}^{\infty} h^{1-s} \log^s h \cdot \psi^s(h) < \infty. \tag{6}$$

Therefore without loss of generality we can assume that

$$q^{1-\frac{2}{s}}(\log q)^{-2-\frac{1}{s}} < \psi(q) \tag{7}$$

for sufficiently large q. To see why, suppose that (7) is not satisfied. Then we replace ψ with the auxiliary function

$$\widetilde{\psi} : q \mapsto \widetilde{\psi} := \max\{\psi(q), q^{1-\frac{2}{s}}(\log q)^{-2-\frac{1}{s}}\}.$$

Clearly, $\widetilde{\psi}$ is an approximation function. One can easily check that (6) and (7) are satisfied with ψ replaced by $\widetilde{\psi}$. Furthermore,

$$S_2^*(\widetilde{\psi}) \supset S_2^*(\psi).$$

Thus it suffices to prove the theorem with ψ replaced by $\widetilde{\psi}$ and (7) can be assumed.

The set $C_f \cap S_2^*(\psi)$ is a $\limsup$-set with the following natural representation:

$$C_f \cap S_2^*(\psi) = \bigcap_{n=1}^{\infty} \bigcup_{q=n}^{\infty} \bigcup_{(p_1,p_2)\in\mathbb{Z}^2} S^*(p_1,p_2,q)$$

where

$$S^*(p_1,p_2,q) := \left\{(x,y) \in C_f \; : \; \left|x - \frac{p_1}{q}\right| \cdot \left|y - \frac{p_2}{q}\right| < \frac{\psi(q)}{q^2}\right\}.$$

Using the fact that ψ is decreasing, we have that for any n

$$C_f \cap S_2^*(\psi) \subset \bigcup_{t=n}^{\infty} \bigcup_{2^t \leqslant q < 2^{t+1}} \bigcup_{(p_1,p_2)\in\mathbb{Z}^2} S^*(p_1,p_2,q,t) \tag{8}$$

where

$$S^*(p_1,p_2,q,t) := \left\{(x,y) \in C_f \; : \; \left|x - \frac{p_1}{q}\right| \cdot \left|y - \frac{p_2}{q}\right| < \frac{\psi(2^t)}{(2^t)^2}\right\}.$$

If $t \in \mathbb{N}, (x,y) \in C_f, q \in \mathbb{N}$ with $2^t \leqslant q < 2^{t+1}$ and

$$\left|x - \frac{p_1}{q}\right| \cdot \left|y - \frac{p_2}{q}\right| < \frac{\psi(2^t)}{(2^t)^2}$$

for some $(p_1,p_2) \in \mathbb{Z}^2$, then there is a unique integer m such that

$$2^{m-1}\frac{\sqrt{2\psi(2^t)}}{2^t} \leqslant \left|x - \frac{p_1}{q}\right| < 2^m\frac{\sqrt{2\psi(2^t)}}{2^t}.$$

For this number m, it follows that

$$\left|y - \frac{p_2}{q}\right| < 2^{-m}\frac{\sqrt{2\psi(2^t)}}{2^t}.$$

Let

$$\gamma_t := \frac{\sqrt{2\psi(2^t)}}{2^t}.$$

Then,

$$C_f \cap S_2^*(\psi) \subset \bigcup_{t=n}^{\infty} \bigcup_{2^t \leqslant q < 2^{t+1}} \bigcup_{(p_1,p_2) \in \mathbb{Z}^2} \bigcup_{m=-\infty}^{+\infty} C_f \cap S(q,p_1,p_2,m) \qquad (9)$$

where

$$S(q,p_1,p_2,m) = \left\{ (x,y) \in \mathbb{R}^2 \ : \ \left| x - \frac{p_1}{q} \right| < 2^m \gamma_t, \ \left| y - \frac{p_2}{q} \right| < 2^{-m} \gamma_t \right\}.$$

Thus, we have constructed a sequence of coverings of $C_f \cap S_2^*(\psi)$. The aim is now to show that if, for a given s, (6) holds than the associated sequence of Hausdorff s-measures for these coverings tends to 0 as $n \to \infty$. It then follows that $\mathcal{H}^s(C_f \cap S_2^*(\psi) = 0$, as required.

To proceed we consider two separate cases. For a fixed t, Case (a): $m \in \mathbb{Z}$ such that

$$2^{-|m|} \geqslant t\sqrt{\psi(2^t)}. \qquad (10)$$

and Case (b): $m \in \mathbb{Z}$ such that

$$2^{-|m|} \leqslant t\sqrt{\psi(2^t)}. \qquad (11)$$

Case (a): First, observe that (10) together with (7) implies that

$$2^{-|m|} \geqslant t\sqrt{2^{t(1-\frac{2}{s})} \cdot t^{-2-\frac{1}{s}}} \Rightarrow 2^{|m|} \leqslant t^{\frac{1}{s}} 2^{t(\frac{1}{s}-\frac{1}{2})}.$$

Upon taking logarithms of both sides of the above inequality, we arrive at

$$|m| \leqslant t\left(\frac{2-s}{2s}\right) + \frac{1}{2s}\log t \ll t. \qquad (12)$$

As $f'(x) > c_1$ for all $x \in I$ it follows that

$$\operatorname{diam}(C_f \cap S(q,p_1,p_2,m)) \ll 2^{-|m|}\frac{\sqrt{\psi(2^t)}}{2^t}. \qquad (13)$$

The implied constant depends on only c_1 and is irrelevant to the remainder of the argument.

Given t and m, let $N(t,m)$ denote the number of triples (q,p_1,p_2) with $2^t \leqslant q < 2^{t+1}$ such that $C_f \cap S(q,p_1,p_2,m) \neq \emptyset$. Suppose now that $C_f \cap S(q,p_1,p_2,m) \neq \emptyset$. Then for some $(x,y) \in C_f$ and θ_1, θ_2 satisfying $-1 < \theta_1, \theta_2 < 1$, we have that

$$x = \frac{p_1}{q} + \theta_1 2^{|m|}\frac{\sqrt{2\psi(2^t)}}{2^t}, \quad y = \frac{p_2}{q} + \theta_2 2^{|m|}\frac{\sqrt{2\psi(2^t)}}{2^t}.$$

Thus, it can be shown that

$$f\left(\frac{p_1}{q}\right) - \frac{p_2}{q} = f\left(\frac{p_1}{q}\right) - f(x) + f(x) - y + y - \frac{p_2}{q}$$

$$= -\theta_2 f'(\xi) \cdot 2^{|m|}\frac{\sqrt{2\psi(2^t)}}{2^t} + \theta_1 \cdot 2^{|m|}\frac{\sqrt{2\psi(2^t)}}{2^t}$$

where ξ lies between x and p_1/q. Further, one can easily deduce that

$$\left| f\left(\frac{p_1}{q}\right) - \frac{p_2}{q} \right| \ll 2^{|m|} \frac{\sqrt{\psi(2^t)}}{2^t} \leqslant \frac{1}{t2^t}.$$

Set $Q = 2^{t+1}$. Then we have $q \leqslant Q$ and

$$\left| f\left(\frac{p_1}{q}\right) - \frac{p_2}{q} \right| \ll \frac{1}{Q\log Q}.$$

As mentioned in the introductory section of this article, a result of Vaughan & Velani is crucial to our argument. We now state this result, which is Theorem 2 from [15].

Theorem VV. *Let $N_f(Q,\psi,I) = \#\{\mathbf{p}/q \; : \; q \leqslant Q, p_1/q \in I, |f(p_1/q) - p_2/q| < \psi(Q)/Q\}$. Suppose that ψ is an approximating function with $\psi(Q) \geqslant Q^{-\phi}$ where ϕ is any real number with $\phi \leqslant \frac{2}{3}$. Then*

$$N_f(q,\psi,I) \ll \psi(Q)Q^2. \tag{14}$$

In our case $\psi(Q) = 2^{|m|}\sqrt{2\psi(2^t)} \asymp \frac{1}{\log Q}$ which satisfies the conditions of Theorem VV. Therefore there exists an absolute constant $c > 0$ such that

$$N(t,m) \ll 2^{2t}2^{|m|}\sqrt{\psi(2^t)}. \tag{15}$$

Now using (13) and (15) we can bound the Hausdorff sum associated with the set $C_f \cap S_2^*(\psi)$

$$\mathcal{H}^s(C_f \cap S_2^*(\psi)) \ll \sum_{t=n}^{\infty} \sum_{m \in \text{case (a)}} \left(2^{-|m|} \frac{\sqrt{\psi(2^t)}}{2^t} \right)^s \times 2^{|m|}2^{2t}\sqrt{\psi(2^t)}$$

$$\ll \sum_{t=n}^{\infty} \sum_{m \in \text{case (a)}} \psi(2^t)^{\frac{1}{2}(1+s)} \cdot 2^{t(2-s)} \cdot 2^{|m|(1-s)}$$

$$\overset{(9)}{\ll} \sum_{t=n}^{\infty} \sum_{m \in \text{case (a)}} t^{s-1}\psi(2^t)^s 2^{t(2-s)}$$

$$\overset{(12)}{\ll} \sum_{t=n}^{\infty} t^s 2^{t(2-s)}\psi(2^t)^s \asymp \sum_{q=2^n}^{\infty} q^{1-s}\log^s q \cdot \psi(q)^s.$$

The above comparability follows from the fact that ψ is an approximating function and therefore decreasing. In view of (6),

$$\sum_{q=2^n}^{\infty} q^{1-s}\log^s q \cdot \psi^s(q) \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore, for Case (a) it follows that $\mathcal{H}^s(C_f \cap S_2^*(\psi)) = 0$ as required.

Case (b): In view of (11), we have that

$$S(q, p_1, p_2, m) \subset \big(S'(q, p_1) \times [0, 1]\big) \cup \big([0, 1] \times S'(q, p_2)\big)$$

where

$$S'(q, p) = \left\{ y \in [0, 1] \ : \ \left|y - \frac{p}{q}\right| < \frac{2t\psi(2^t)}{2^t} \right\}.$$

Thus, the set of (9) is a subset of

$$\bigcup_{t=n}^{\infty} \bigcup_{2^t \leqslant q < 2^{t+1}} \bigcup_{(p_1, p_2) \in \mathbb{Z}^2} C_f \cap \big(\big(S'(q, p_1) \times [0, 1]\big) \cup \big([0, 1] \times S'(q, p_2)\big)\big). \quad (16)$$

As $c_1 > f'(x) > c_2 > 0$, for any choice of p_1, p_2 and q appearing in (16),

$$\mathrm{diam}(C_f \cap (S'(q, p_1) \times [0, 1])) \ll t\frac{\psi(2^t)}{2^t}$$

and

$$\mathrm{diam}(C_f \cap ([0, 1] \times S'(q, p_2))) \ll t\frac{\psi(2^t)}{2^t}.$$

The implied constants depends only on c_1 and c_2 and are irrelevant in the context of the rest of the proof. Furthermore, for a fixed t and q in (16), the number of $(p_1, p_2) \in \mathbb{Z}^2$ for which

$$C_f \cap \big(\big(S'(q, p_1) \times [0, 1]\big) \cup \big([0, 1] \times S'(q, p_2)\big)\big)$$

are non-empty and disjoint is $\ll q$. Drawing all the above considerations together it follows that the Hausdorff s-sum for this covering of the set $C_f \cap S_2^*(\psi)$, as defined in (16), is bounded above by

$$\sum_{t=n}^{\infty} \left(t\frac{\psi(2^t)}{2^t}\right)^s 2^{2t} \asymp \sum_{q=2^n}^{\infty} q^{1-s} \log^s q \cdot \psi^s(q) \to 0 \ \text{ as } n \to \infty.$$

As in the previous case, Case (a), by letting $n \to \infty$ we conclude that

$$\mathcal{H}^s(C_f \cap S_2^*(\psi)) = 0$$

This completes the proof of Theorem 1.

4. Proof of Theorem 2

We now proceed with establishing Theorem 2. The proof is somewhat analogous to that of Theorem 1 and for brevity we leave the technical details needed to verify some of our estimates to the reader. Especially in those cases when the estimates following in exactly the same manner or require only minor modifications to the arguments used in the proof Theorem 1.

For the sake of convenience, let $\psi = \psi_1$ and $\phi = \psi_2$. It is clear that

$$S_2(\phi, \psi) \subset S_2(\psi^*, \psi_*) \cup S_2(\psi_*, \psi^*)$$

where

$$\psi_* = \min\{\psi, \phi\} \quad \& \quad \psi^* = \max\{\psi, \phi\}.$$

Since $\psi^* \psi_* = \psi\phi$, we have that $\sum \psi^*(q)\psi_*(q) < \infty$. Thus to prove Theorem 2 it is sufficient to show that both the sets $C_f \cap S_2(\psi^*, \psi_*)$ and $C_f \cap S_2(\psi_*, \psi^*)$ are of Lebesgue measure zero. We will consider one of these two sets, the other case is similar. Thus, without any loss of generality we assume that $\psi(q) \geqslant \phi(q)$ for all $q \in \mathbb{N}$.

Since $\sum_{q=1}^{\infty} \psi(q)\phi(q) < \infty$ and both ψ, ϕ are decreasing we have that $\psi(q)\phi(q) < q^{-1}$ for all sufficiently large q. Hence, $\phi(q) \leqslant q^{-1/2}$ for sufficiently large q. Further, we can assume that

$$\psi(q) \geqslant q^{-2/3} \tag{17}$$

for all $q \in \mathbb{N}$. To see this consider the auxiliary function $\tilde{\psi}$ where

$$\tilde{\psi}(q) = \max\{\psi(q), q^{-2/3}\}.$$

Clearly, $\tilde{\psi}$ is an approximating function. It also satisfies the following set inclusion,

$$S_2(\psi, \phi) \subset S_2(\tilde{\psi}, \phi).$$

Moreover,

$$\sum_{q=1}^{\infty} \tilde{\psi}(q)\phi(q) \leqslant \sum_{q=1}^{\infty} \psi(q)\phi(q) + \sum_{q=1}^{\infty} q^{-2/3}\phi(q)$$

$$\ll \sum_{q=1}^{\infty} \psi(q)\phi(q) + \sum_{q=1}^{\infty} q^{-2/3}q^{-1/2} < \infty.$$

This means that it is sufficient to prove Theorem 2 with ψ replaced by $\tilde{\psi}$ and therefore without any loss of generality, (17) can be assumed.

In a manner analogous to that of (8), it is readily verified that for any $n \geqslant 1$

$$C_f \cap S_2(\psi, \phi) \subset \bigcup_{t=n}^{\infty} \bigcup_{2^t \leqslant q < 2^{t+1}} \bigcup_{(p_1, p_2) \in \mathbb{Z}^2} C_f \cap S_2(p_1, p_2, q) \tag{18}$$

where

$$S_2(p_1, p_2, q) = \left\{ (x, y) \in \mathbb{R}^2 : \left| x - \frac{p_1}{q} \right| < \frac{\psi(2^t)}{2^t}, \left| y - \frac{p_2}{q} \right| < \frac{\phi(2^t)}{2^t} \right\}$$

and t is uniquely defined by $2^t \leqslant q < 2^{t+1}$. Next, we can use the same argument to that used in (13) to verify that

$$|C_f \cap S_2(q, p_1, p_2)|_{C_f} \ll \frac{\phi(2^t)}{2^t}. \tag{19}$$

Finally, for fixed t let $N(t)$ be the number of triples (q, p_1, p_2) with $2^t \leqslant q < 2^{t+1}$ such that $C_f \cap S(q, p_1, p_2) \neq \emptyset$. On modifying the argument used to establish (15), one obtains the estimate

$$N(t) \ll 2^{2t}\psi(2^t). \tag{20}$$

The upshot of (18), (19) and (20) is that

$$|C_f \cup S_2(\psi, \phi)|_{C_f} \ll \sum_{t=n}^{\infty} \sum_{2^t \leqslant q < 2^{t+1}} \bigcup_{(p_1, p_2) \in \mathbb{Z}^2} C_f \cap S_2(q, p_1, p_2)$$

$$\ll \sum_{t=n}^{\infty} N(t) \frac{\phi(2^t)}{2^t} \ll \sum_{t=n}^{\infty} 2^t \psi(2^t) \phi(2^t) \asymp \sum_{q=2^n}^{\infty} \psi(q) \phi(q).$$

Since $\sum_{q=1}^{\infty} \psi(q)\phi(q) < \infty$, we have that $\sum_{q=2^n}^{\infty} \psi(q)\phi(q) \to 0$ as $n \to \infty$ and it follows that

$$|C_f \cap S_2(\psi, \phi)|_{C_f} = 0$$

as required.

This completes the proof of Theorem 2.

5. Remarks and Possible Developments

An obvious next step is to establish the divergence counterpart to Theorem 1. That is, one would like to show that

$$\sum_{h=1}^{\infty} h^{1-s} (\log^s h) \psi^s(h) = \infty \implies \mathcal{H}^s(C_f \cap S_2^*(\psi)) = \infty.$$

By adapting the arguments in this paper and using the ideas of local ubiquity, as developed in [2], it is likely that one could establish a zero-full result for $\mathcal{H}^h(C_f \cap S_2^*(\psi))$ where h is a general dimension function. This would include the above result and Theorem 1 as a special case. A dimension function $h : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing, continuous function such that $h(r) \to 0$ as $r \to 0$. By replacing the quantity $\text{diam}^s(C_i)$ with $h(\text{diam}(C_i))$ in the definition of $\mathcal{H}^s$, one can define the Hausdorff h-measure of a set. For further details see [9] or [12]. Dimension functions give very precise information about the measure theoretic properties of a set. The convergence part of such a theorem follows almost immediately on from Theorem 1. Most of the estimates obtained in the proof of Theorem 1 remain the same, the generalisation to Hausdorff h-measures effects only the estimates involving the measures of the actual covers defined in the proof. The main task in proving such a theorem would be in the proof of the divergence case.

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